Kompleksni brojevi

1. Odrediti sve kompl. br. z takve da brojevi $z, \frac{1}{z}, 1-z$ imaju jednake module. Za tako nađeno z izračunati $\sqrt[3]{(z+\frac{1}{z}+i)^5}$

1) $|z| = \left|\frac{1}{z}\right|$

$|z| = \frac{1}{|z|}$

$|z|^2 = 1$

$|z| = 1$

2) $|z| = |1-z| \Rightarrow |1-z| = 1$

$z = x + yi$

$|z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$

$x^2+y^2 = 1$

$1-z = 1-x-yi$

$|1-z| = 1 \Rightarrow \sqrt{(1-x)^2+(-y)^2} = 1$

$(1-x)^2+y^2 = 1$

3) $\begin{cases} x^2+y^2 = 1 \\ \underline{(1-x^2)+y^2 = 1} \end{cases} \Big\rangle (-1)$

$\begin{cases} x^2+y^2 = 1 \\ 1-2x = 0 \Rightarrow x = \frac{1}{2} \end{cases}$

$y^2 = 1-x^2$

$y^2 = \frac{3}{4}$

$y = \pm\frac{\sqrt{3}}{2}$

$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

4) $z + \frac{1}{z} + i = z + \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} + i =$

$= z + \frac{\bar{z}}{|z|^2} + i = z + \bar{z} + i =$

$= x + \cancel{yi} + x - \cancel{yi} + i = 2x + i =$

$= 2\cdot\frac{1}{2} + i = 1 + i$

$t = 1+i$

$t = |t|(\cos\varphi + i\sin\varphi)$

$a = 1,\ b = 1$

$|t| = \sqrt{a^2+b^2} = \sqrt{2}$

$\operatorname{tg}\varphi = \frac{b}{a}$

$\operatorname{tg}\varphi = 1 \Rightarrow \varphi = \frac{\pi}{4}$

$t = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

5) $t^5 = (\sqrt{2})^5 \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right) \quad k=0,1,\dots n-1$$

$$\sqrt[3]{t^5} = \sqrt[3]{(\sqrt{2})^5}\left(\cos\frac{\frac{5\pi}{4}+2k\pi}{3} + i\sin\frac{\frac{5\pi}{4}+2k\pi}{3}\right), \quad k=0,1,2$$

$k=0: \quad z_0 = \sqrt[3]{4\sqrt{2}}\cdot\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

$k=1: \quad z_1 = \sqrt[3]{4\sqrt{2}}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

$k=2: \quad z_2 = \sqrt[3]{4\sqrt{2}}\left(\cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}\right)$

2. Neka je $\operatorname{Im}\left(\frac{z+2}{z-i}\right) = 1$ i $\operatorname{Re}(z^2+1) = 1$. Naći z

$z = x + yi$

$$\frac{z+2}{z-i} = \frac{x+yi+2}{x+yi-i} = \frac{x+2+yi}{x+(y-1)i}\cdot\frac{x-(y-1)i}{x-(y-1)i} = \frac{(x+2)x - (x+2)(y-1)i + xyi - y(y-1)i^2}{x^2-(y-1)^2\cdot(-1)}$$

$$= \frac{x^2+2x+y^2-y+(\cancel{-xy}+x-2y+2+\cancel{xy})i}{x^2+(y-1)^2} =$$

$$\frac{z+2}{z-i} = \frac{x^2+2x+y^2-y}{x^2+y^2-2y+1} + \frac{x-2y+2}{x^2+y^2-2y+1}i$$

$$\operatorname{Im}\left(\frac{z+2}{z-i}\right) = 1 \Rightarrow \frac{x-2y+2}{x^2+y^2-2y+1} = 1$$

$$z^2+1 = (x+yi)^2+1 = x^2+2xyi+y^2\cdot(-1)+1 = x^2-y^2+1+2xyi$$

$$\operatorname{Re}(z^2+1) = 1 \Rightarrow x^2-y^2+\cancel{1} = \cancel{1}$$

$$x^2 = y^2$$

$$x = \pm y$$

$1^\circ \quad x = y$

$$\frac{x - 2x + 2}{x^2 + x^2 - 2x + 1} = 1$$

$$\frac{2 - x}{2x^2 - 2x + 1} = 1$$

$$2 - x = 2x^2 - 2x + 1$$

$$2x^2 - x - 1 = 0$$

$$x_{1,2} = \frac{1 \pm 3}{4}$$

$$x_1 = 1 \quad x_2 = -\frac{1}{2}$$

$$z_1 = 1 + i$$

$$z_2 = -\frac{1}{2} - \frac{1}{2} i$$

$2^\circ \quad x = -y \Rightarrow y = -x$

$$\frac{x + 2x + 2}{x^2 + x^2 + 2x + 1} = 1$$

$$\frac{3x + 2}{2x^2 + 2x + 1} = 1$$

$$3x + 2 = 2x^2 + 2x + 1$$

$$2x^2 - x - 1 = 0$$

$$x_3 = 1 \quad , \quad x_4 = -\frac{1}{2}$$

$$z_3 = 1 - i$$

$$z_4 = -\frac{1}{2} + \frac{1}{2} i$$

17. oktobar

1. Neka je $|z_1| = 2$, $\arg z_1 = \frac{2\pi}{3}$, $\arg z_2 = \frac{\pi}{6}$, $\arg z_3 = -\frac{\pi}{3}$, $z_1 + z_2 + z_3 =$
Naći z_1, z_2, z_3 i izračunati $\frac{z_2^3 \cdot z_3^4}{z_1^2}$.

$$z_1 = |z_1| \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$z_1 = 2 \left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \right)$$

$$z_1 = -1 + i\sqrt{3}$$

$$z_2 = |z_2| \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$z_2 = |z_2| \left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right)$$

$$z_2 = |z_2| \cdot \frac{\sqrt{3}}{2} + \frac{|z_2|}{2} \cdot i$$

$$z_3 = |z_3| \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right)$$

$$z_3 = |z_3| \cdot \left(\frac{1}{2} + i \cdot \left(-\frac{\sqrt{3}}{2}\right) \right)$$

$$z_3 = \frac{|z_3|}{2} - \frac{\sqrt{3}}{2} |z_3| \cdot i$$

$$-1 + i\sqrt{3} + |z_2|\cdot\frac{\sqrt{3}}{2} - \frac{|z_2|}{2}i + \frac{|z_3|}{2} - \frac{\sqrt{3}}{2}|z_3|\cdot i = 1$$

$$-1 + \frac{|z_2|}{2}\cdot\sqrt{3} + \frac{|z_3|}{2} + \left(\sqrt{3} - \frac{|z_2|}{2} - \frac{\sqrt{3}}{2}|z_3|\right)i = 1 + 0\cdot i$$

$$\begin{cases} -1 + \frac{|z_2|}{2}\sqrt{3} + \frac{|z_3|}{2} = 1 \\ \sqrt{3} - \frac{|z_2|}{2} - \frac{\sqrt{3}}{2}|z_3| = 0 \end{cases} \qquad |z_2| = \sqrt{3}, \ |z_3| = 1$$

$$z_2 = \frac{3}{2} + \frac{\sqrt{3}}{2}i$$

$$z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad z_2 = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \qquad z_3 = 1\cdot\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z_1^2 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \qquad z_2^3 = 3\sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \qquad z_3^4 = 1\cdot\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$$

$$\frac{z_2^3\cdot z_3^4}{z_1^2} = \frac{3\sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot 1\cdot\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)}{4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)} = \frac{3\sqrt{3}\left(\cos\left(\frac{\pi}{2} - \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{2} - \frac{4\pi}{3}\right)\right)}{4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}$$

$$= \frac{3\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)}{4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)} = \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{5\pi}{6} - \frac{4\pi}{3}\right) + i\sin\left(-\frac{5\pi}{6} - \frac{4\pi}{3}\right)\right) =$$

$$= \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{13\pi}{6}\right) + i\sin\left(-\frac{13\pi}{6}\right)\right) = \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) =$$

$$= \frac{3\sqrt{3}}{4}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \frac{3\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = \frac{9}{8} - \frac{3\sqrt{3}}{8}i$$

2. Izračunati $(z-2i)^5$ ako je $|\bar{z}+i| = \sqrt{3}$ i $\arg(\bar{z}+i) = \pi$.

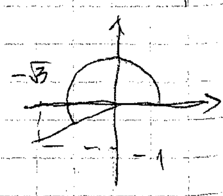
$$\bar{z} + i = \sqrt{3}\left(\cos\pi + i\sin\pi\right) \qquad t = z - 2i = -\sqrt{3} + i - 2i = -\sqrt{3} - i$$

$$\bar{z} + i = \sqrt{3}\left(-1 + i\cdot 0\right) \qquad t = |t|\left(\cos\varphi + i\sin\varphi\right)$$

$$\bar{z} + i = -\sqrt{3} \qquad |t| = \sqrt{3+1} = 2$$

$$\bar{z} = -\sqrt{3} - i$$

$$z = -\sqrt{3} + i \qquad \operatorname{tg}\varphi = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$



$$\varphi = \frac{7\pi}{6}$$

$$t = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$t^5 = 2^5\left(\cos\frac{35\pi}{6} + i\sin\frac{35\pi}{6}\right)$$

$$t^5 = 32\left(\cos\left(4\pi + \frac{11\pi}{6}\right) + i\sin\left(4\pi + \frac{11\pi}{6}\right)\right)$$

$$t^5 = 32\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$t^5 = 32\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$

$$t^5 = 16\sqrt{3} - 16i$$

3. Odrediti kompleksne brojeve z i $\sqrt{w}$ ako je $\arg z = \frac{\pi}{4}$, $\arg \bar{w} = \frac{\pi}{6}$, $z + \bar{w} = \sqrt{3} + i$.

$$z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \bar{w} = |\bar{w}|\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z = |z|\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \qquad \bar{w} = |\bar{w}|\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

$$z = |z|\frac{\sqrt{2}}{2} + |z|\frac{\sqrt{2}}{2}\cdot i \qquad \bar{w} = \frac{|\bar{w}|\sqrt{3}}{2} + \frac{|\bar{w}|}{2}i$$

$$z + \bar{w} = \sqrt{3} + i$$

$$\frac{|z|\sqrt{2}}{2} + \frac{|z|\sqrt{2}}{2}i + \frac{|\bar{w}|\sqrt{3}}{2} + \frac{|\bar{w}|}{2}i = \sqrt{3} + i$$

$$\begin{cases} \frac{|z|\sqrt{2}}{2} + \frac{|\bar{w}|\sqrt{3}}{2} = \sqrt{3} \\ \frac{|z|\sqrt{2}}{2} + \frac{|\bar{w}|}{2} = 1 \end{cases} \Rightarrow |z| = 0 \quad \boxed{|\bar{w}| = 2 \Downarrow |w| = 2} \Rightarrow \bar{w} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$|z| = 0 \Downarrow \boxed{z = 0}$$

$$\bar{w} = 2\cdot\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\underline{\bar{w} = \sqrt{3} + i}$$

$$\boxed{\arg w = 2\pi - \arg \bar{w}} \qquad \underline{w = \sqrt{3} - i}$$

$$\arg w = 2\pi - \frac{\pi}{6}$$

$$\arg w = \frac{11\pi}{6}$$

$$w = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$\sqrt{w} = \sqrt{2}\left(\cos\frac{\frac{11\pi}{6} + 2k\pi}{2} + i\sin\frac{\frac{11\pi}{6} + 2k\pi}{2}\right)$$

$$\boxed{\begin{array}{l} k = 0, \quad w_0 = \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \\ k = 1, \quad w_1 = \sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) \end{array}}$$

4. Dat je broj $z = \left[\frac{i^{34} - i^{71}}{2} + \frac{2}{(1-i)^3} \right]^6$. Odrediti w ako važe jednakosti $\arg \bar{w} = \frac{7\pi}{4}$ i $\operatorname{Im}(z-w) = 4$.

$$\arg w = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$$

$$i^{34} = i^{4\cdot 8+2} = -1$$

$$i^{71} = i^{4\cdot 17+3} = -i$$

$$(1-i)^3 = 1 - 3\cdot i + 3\cdot 1\cdot i^2 - i^3 =$$

$$= 1 - 3i - 3 + i = -2 - 2i$$

$$\frac{i^{34} - i^{71}}{2} + \frac{2}{(1-i)^3} = \frac{-1-(-i)}{2} + \frac{2}{-2-2i} = \frac{i-1}{2} - \frac{1}{1+i} \cdot \frac{i-1}{i-1} =$$

$$= \frac{i-1}{2} + \frac{i-1}{2} = i - 1$$

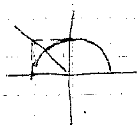
$\underline{(i-1)^6}$

$$t = i - 1$$

$$t = -1 + i$$

$$t = |t|(\cos\varphi + i\sin\varphi)$$

$$\operatorname{tg}\varphi = \frac{1}{-1} = -1 \Rightarrow \varphi = \frac{3\pi}{4}$$



$$|t| = \sqrt{1+1} = \sqrt{2}$$

$$t = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$t^6 = 8\left(\cos\frac{18\pi}{4} + i\sin\frac{18\pi}{4}\right)$$

$$t^6 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$t^6 = 8\cdot(0 + i\cdot 1)$$

$$t^6 = 8i$$

$$z = 8i$$

$$w = |w|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w = \frac{|w|\sqrt{2}}{2} + \frac{|w|\sqrt{2}}{2} i$$

$$z - w = 8i - \frac{|w|\sqrt{2}}{2} - \frac{|w|\sqrt{2}}{2} i =$$

$$= -\frac{|w|\sqrt{2}}{2} + \left(8 - \frac{|w|\sqrt{2}}{2}\right) i$$

$$\operatorname{Im}(z-w) = 4$$

$$8 - \frac{|w|\sqrt{2}}{2} = 4$$

$$\frac{|w|\sqrt{2}}{2} = 4$$

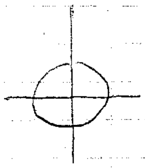
$$|w| = 4\sqrt{2}$$

$$\boxed{w = 4 + 4i}$$

5. Naći kompleksni broj z koji je zadat sa jednačinama $Re\left(z^2+1-Re\left(\frac{1-i}{i}\right)\right)=1$ i $\arg\left(z^3\cdot(1+i\sqrt{3})\right)=\frac{5\pi}{6}$

$t = 1+\sqrt{3}\,i$

$|t| = \sqrt{1+3} = 2$



$tg\,\varphi = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\varphi = \frac{\pi}{3}$

$t = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z = |z|\left(\cos\alpha + i\sin\alpha\right)$

!!

$z^3 = |z|^3\left(\cos 3\alpha + i\sin 3\alpha\right)$

$z^3\cdot t = 2|z|^3\left(\cos\left(\frac{\pi}{3}+3\alpha\right) + i\sin\left(\frac{\pi}{3}+3\alpha\right)\right)$

$\arg(z^3\cdot t) = \frac{5\pi}{6} \implies \frac{\pi}{3}+3\alpha = 2k\pi + \frac{5\pi}{6}$

$3\alpha = \frac{5\pi}{6} - \frac{\pi}{3} + 2k\pi$

$3\alpha = \frac{\pi}{2} + 2k\pi$

$\alpha = \frac{\pi}{6} + \frac{2k\pi}{3}$

$k=0 \Rightarrow \alpha = \frac{\pi}{6}$

$k=1 \Rightarrow \alpha = \frac{5\pi}{6}$

$k=2 \Rightarrow \alpha = \frac{9\pi}{6} = \frac{3\pi}{2}$

$k=3 \Rightarrow \alpha = \frac{13\pi}{6} > 2\pi$

$\frac{1-i}{i}\cdot\frac{i}{i} = \frac{i-(-1)}{-1} = \frac{i+1}{-1} = -1-i$

$Re\left(\frac{1-i}{i}\right) = -1$

$z^2 = |z|^2\left(\cos 2\alpha + i\sin 2\alpha\right)$

$z^2 = |z|^2\cos 2\alpha + |z|^2\sin 2\alpha\cdot i$

$Re\left(|z|^2\cos 2\alpha + |z|^2\sin 2\alpha\cdot i + 1-(-1)\right) = 1$

$Re\left(2 + |z|^2\cos 2\alpha + |z|^2\sin 2\alpha\cdot i\right) = 1$

$2 + |z|^2\cos 2\alpha = 1$

$|z|^2 = \frac{-1}{\cos 2\alpha} \Rightarrow \cos 2\alpha < 0$

$\alpha = \frac{\pi}{6} \quad |z|^2 = \frac{-1}{\cos\frac{\pi}{3}} = \frac{-1}{\frac{1}{2}}$

$\alpha = \frac{5\pi}{6} \quad |z|^2 = \frac{-1}{\cos\frac{5\pi}{3}} = \frac{-1}{\frac{1}{2}}$

$\alpha = \frac{3\pi}{2} \quad |z|^2 = \frac{-1}{\cos 3\pi} = \frac{-1}{-1} = 1$

$|z| = 1$

$z = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \quad \boxed{z=-i}$

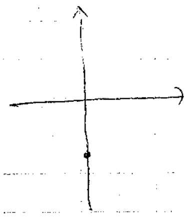
6. Izračunati $\sqrt[4]{z}$ ako je $\frac{z}{\bar{z}} + \frac{\bar{z}}{z} + 4\cdot\left(\frac{1}{z} + \frac{1}{\bar{z}}\right) = 2$ i $\arg(z-2) = -\frac{\pi}{2}$

$z = x + yi$

$z - 2 = x + yi - 2 = x - 2 + yi$

$\arg(z-2) = -\frac{\pi}{2}$

$$\left.\begin{array}{l} z-2 = x-2+yi \\ \arg(z-2) = -\frac{\pi}{2} \end{array}\right\} \begin{array}{l} x-2=0 \Rightarrow x=2 \\ y<0 \end{array}$$



$Re(z-2)=0$

$Im(z-2)<0$

$$\frac{z}{\bar{z}} + \frac{\bar{z}}{z} + 4\left(\frac{1}{z} + \frac{1}{\bar{z}}\right) = 2$$

$$\frac{z^2 + \bar{z}^2}{\bar{z}\cdot z} + 4\,\frac{\bar{z} + z}{z\cdot\bar{z}} = 2$$

$$\frac{z^2 + \bar{z}^2 + 4(\bar{z} + z)}{|z|^2} = 2$$

$$\frac{x^2 + \cancel{2xyi} - y^2 + x^2 - \cancel{2xyi} - y^2 + 4(x - \cancel{yi} + x + \cancel{yi})}{x^2+y^2} = 2$$

$$\cancel{2x^2} - 2y^2 + 8x = \cancel{2x^2} + 2y^2$$

$8x = 4y^2$

$y^2 = 2x$

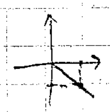
$y^2 = 4$

$y = -2$ jer je $y<0$

$z = 2 - 2i$

$|z| = \sqrt{4+4} = 2\sqrt{2}$

$\operatorname{tg}\varphi = \frac{-2}{2} = -1$



$\varphi = \frac{7\pi}{4}$

$z = 2\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$$\sqrt[4]{z} = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{\frac{7\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{7\pi}{4}+2k\pi}{4}\right)$$

$k = 0, 1, 2, 3$

$k=0 \Rightarrow z_0 =$